Statistical Analysis

Lecture 07

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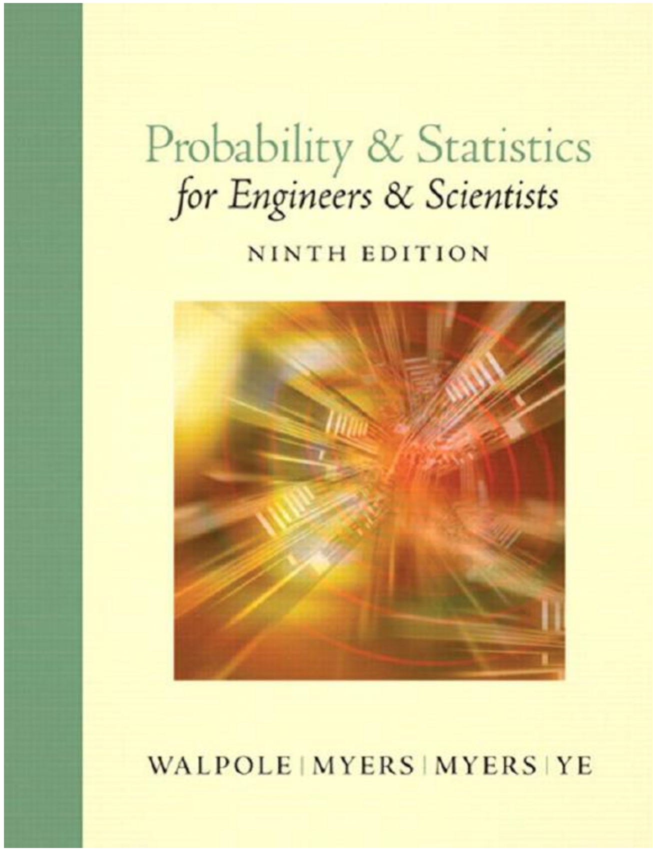
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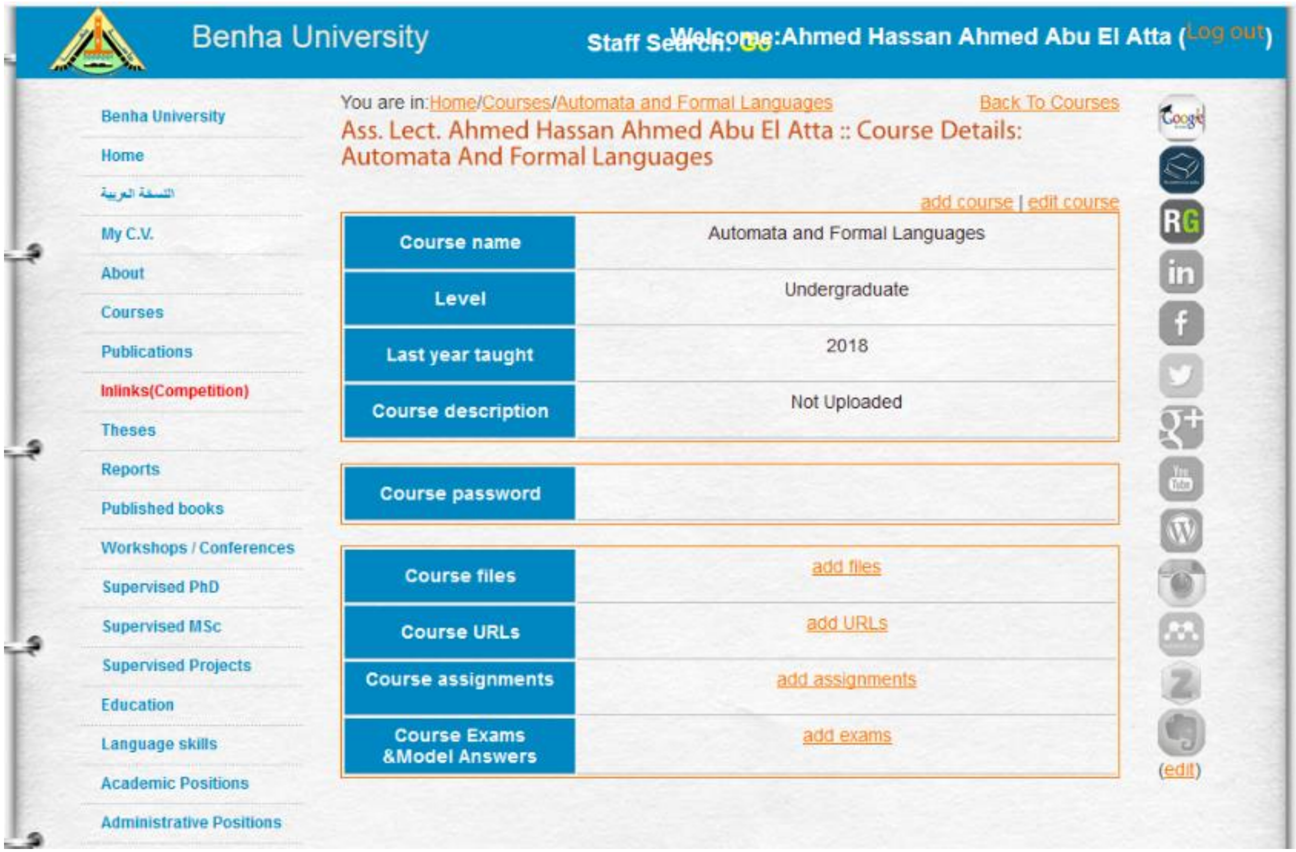

One- and Two Sample Tests of Hypotheses

CHAPTER 10

The Null and Alternative Hypotheses

The Null and Alternative Hypotheses

The structure of hypothesis testing will be formulated with the use of the term

null hypothesis, which refers to any hypothesis we wish to test and is denoted by H_0.

The rejection of H_0 leads to the acceptance of an ***alternative hypothesis***, denoted by H_1.

Example

Test if ***p*** exceeds ***0.10***. We may then state

$$H_0\colon\ p = 0.10,$$
$$H_1\colon\ p > 0.10.$$

One- and Two-Tailed Tests

A test of any statistical hypothesis where the alternative is **one sided**, such as

$$H_0\colon\ \theta = \theta_0,$$
$$H_1\colon\ \theta > \theta_0$$

or perhaps

$$H_0\colon\ \theta = \theta_0,$$
$$H_1\colon\ \theta < \theta_0,$$

One- and Two-Tailed Tests

A test of any statistical hypothesis where the alternative is **two sided**, such as

$$H_0\colon\ \theta = \theta_0,$$
$$H_1\colon\ \theta \neq \theta_0,$$

Example 10.1:

A manufacturer of a certain brand of rice cereal claims that the average saturated fat content does not exceed 1.5 grams per serving. State the null and alternative hypotheses to be used in testing this claim and determine where the critical region is located.

Solution :

A manufacturer of a certain brand of rice cereal claims that the average saturated fat content does not exceed 1.5 grams per serving. State the null and alternative hypotheses to be used in testing this claim and determine where the critical region is located.

The manufacturer's claim should be rejected only if μ is greater than 1.5 milligrams and should not be rejected if μ is less than or equal to 1.5 milligrams. We test

$$H_0: \ \mu = 1.5,$$
$$H_1: \ \mu > 1.5.$$

Nonrejection of H_0 does not rule out values less than 1.5 milligrams. Since we have a one-tailed test, the greater than symbol indicates that the critical region lies entirely in the right tail of the distribution of our test statistic $\bar{X}$.

Example 10.2:

A real estate agent claims that 60% of all private residences being built today are 3-bedroom homes. To test this claim, a large sample of new residences is inspected; the proportion of these homes with 3 bedrooms is recorded and used as the test statistic. State the null and alternative hypotheses to be used in this test and determine the location of the critical region.

Solution

A real estate agent claims that 60% of all private residences being built today are 3-bedroom homes. To test this claim, a large sample of new residences is inspected; the proportion of these homes with 3 bedrooms is recorded and used as the test statistic. State the null and alternative hypotheses to be used in this test and determine the location of the critical region.

If the test statistic were substantially higher or lower than $p = 0.6$, we would reject the agent's claim. Hence, we should make the hypothesis

$$H_0\text{: } p = 0.6,$$
$$H_1\text{: } p \neq 0.6.$$

The alternative hypothesis implies a two-tailed test with the critical region divided equally in both tails of the distribution of $\widehat{P}$, our test statistic.

10.4 Single Sample: Tests Concerning a Single Mean

Tests on a Single Mean (Variance Known)

We should first describe the assumptions on which the experiment is based. The model for the underlying situation centers around an experiment with $X_1, X_2, \ldots,$ X_n representing a random sample from a distribution with mean μ and variance $\sigma^2 > 0$. Consider first the hypothesis

$$H_0\colon \mu = \mu_0,$$
$$H_1\colon \mu \neq \mu_0.$$

Tests on a Single Mean (Variance Known)

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}.$$

We know that *under* H_0, that is, if $\mu = \mu_0$, $\sqrt{n}(\bar{X} - \mu_0)/\sigma$ follows an $n(x; 0, 1)$ distribution, and hence the expression

$$P\left(-z_{\alpha/2} < \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = 1 - \alpha$$

Test Procedure for a Single Mean (Variance Known)

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} > z_{\alpha/2} \quad \text{or} \quad z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} < -z_{\alpha/2}$$

If $-z_{\alpha/2} < z < z_{\alpha/2}$, do not reject H_0. Rejection of H_0, of course, implies acceptance of the alternative hypothesis $\mu \neq \mu_0$. With this definition of the critical region, it should be clear that there will be probability α of rejecting H_0 (falling into the critical region) when, indeed, $\mu = \mu_0$.

Test Procedure for a Single Mean (Variance Known)

Although it is easier to understand the critical region written in terms of z, we can write the same critical region in terms of the computed average $\bar{x}$. The following can be written as an identical decision procedure:

$$\textit{reject } H_0 \text{ if } \bar{x} < a \text{ or } \bar{x} > b,$$

where

$$a = \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \qquad b = \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

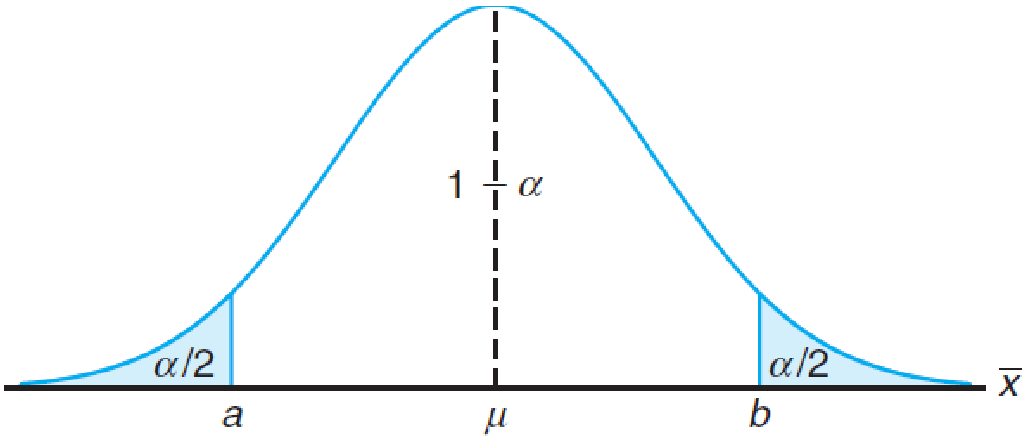

Figure 10.9: Critical region for the alternative hypothesis $\mu \neq \mu_0$.

Tests of one-sided hypotheses on the mean involve the same statistic described in the two-sided case. The difference, of course, is that the critical region is only in one tail of the standard normal distribution. For example, suppose that we seek to test

$$H_0\colon\ \mu = \mu_0,$$
$$H_1\colon\ \mu > \mu_0.$$

The signal that favors H_1 comes from *large values* of z. Thus, rejection of H_0 results when the computed $z > z_\alpha$. Obviously, if the alternative is $H_1\colon \mu < \mu_0$, the critical region is entirely in the lower tail and thus rejection results from $z < -z_\alpha$. Although in a one-sided testing case the null hypothesis can be written as $H_0\colon \mu \leq \mu_0$ or $H_0\colon \mu \geq \mu_0$, it is usually written as $H_0\colon \mu = \mu_0$.

Example 10.3

A random sample of 100 recorded deaths in the United States during the past year showed an average life span of 71.8 years. Assuming a population standard deviation of 8.9 years, does this seem to indicate that the mean life span today is greater than 70 years? Use a 0.05 level of significance.

Solution

1. H_0: $\mu = 70$ years.
2. H_1: $\mu > 70$ years.
3. $\alpha = 0.05$.
4. Critical region: $z > 1.645$, where $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$.
5. Computations: $\bar{x} = 71.8$ years, $\sigma = 8.9$ years, and hence $z = \frac{71.8-70}{8.9/\sqrt{100}} = 2.02$.
6. Decision: Reject H_0 and conclude that the mean life span today is greater than 70 years.

The P-value corresponding to $z = 2.02$ is given by the area of the shaded region in Figure 10.10.

Using Table A.3, we have

$$P = P(Z > 2.02) = 0.0217.$$

As a result, the evidence in favor of H_1 is even stronger than that suggested by a 0.05 level of significance.

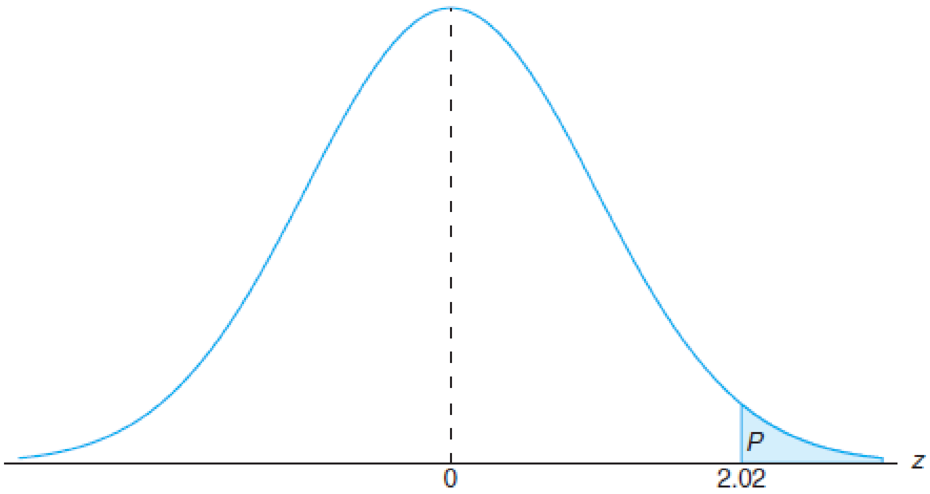

Figure 10.10: P-value for Example 10.3.

Example 10.4

A manufacturer of sports equipment has developed a new synthetic fishing line that the company claims has a mean breaking strength of 8 kilograms with a standard deviation of 0.5 kilogram. Test the hypothesis that $\mu = 8$ kilograms against the alternative that $\mu \neq 8$ kilograms if a random sample of 50 lines is tested and found to have a mean breaking strength of 7.8 kilograms. Use a 0.01 level of significance.

Solution

1. H_0: $\mu = 8$ kilograms.
2. H_1: $\mu \neq 8$ kilograms.
3. $\alpha = 0.01$.
4. Critical region: $z < -2.575$ and $z > 2.575$, where $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$.
5. Computations: $\bar{x} = 7.8$ kilograms, $n = 50$, and hence $z = \frac{7.8-8}{0.5/\sqrt{50}} = -2.83$.
6. Decision: Reject H_0 and conclude that the average breaking strength is not equal to 8 but is, in fact, less than 8 kilograms.

Since the test in this example is two tailed, the desired P-value is twice the area of the shaded region in Figure 10.11 to the left of $z = -2.83$. Therefore, using Table A.3, we have

$$P = P(|Z| > 2.83) = 2P(Z < -2.83) = 0.0046,$$

which allows us to reject the null hypothesis that $\mu = 8$ kilograms at a level of significance smaller than 0.01.

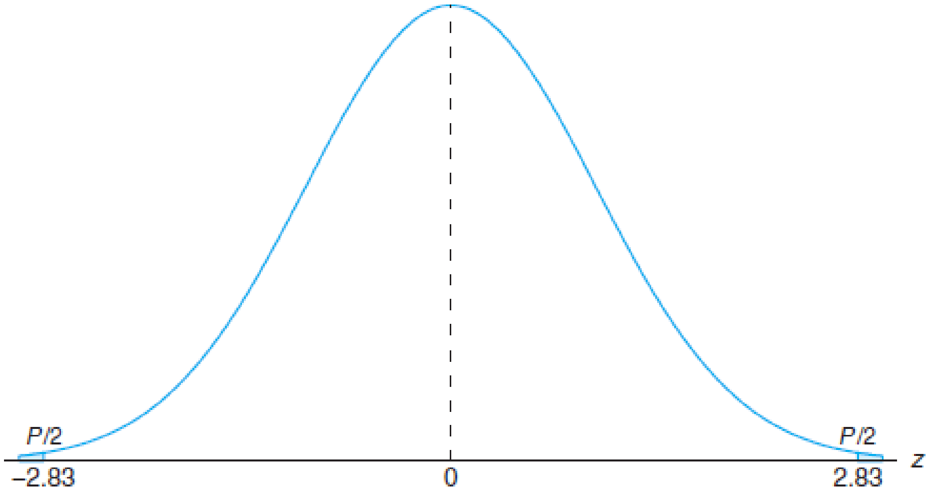

Figure 10.11: P-value for Example 10.4.

Tests on a Single Sample (Variance Unknown)

For the two-sided hypothesis

$$H_0\colon\ \mu = \mu_0,$$
$$H_1\colon\ \mu \neq \mu_0,$$

we reject H_0 at significance level α when the computed t-statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

exceeds $t_{\alpha/2,n-1}$ or is less than $-t_{\alpha/2,n-1}$.

Example 10.5

The Edison Electric Institute has published figures on the number of kilowatt hours used annually by various home appliances. It is claimed that a vacuum cleaner uses an average of 46 kilowatt hours per year. If a random sample of 12 homes included in a planned study indicates that vacuum cleaners use an average of 42 kilowatt hours per year with a standard deviation of 11.9 kilowatt hours, does this suggest at the 0.05 level of significance that vacuum cleaners use, on average, less than 46 kilowatt hours annually? Assume the population of kilowatt hours to be normal.

Solution

1. H_0: $\mu = 46$ kilowatt hours.
2. H_1: $\mu < 46$ kilowatt hours.
3. $\alpha = 0.05$.
4. Critical region: $t < -1.796$, where $t = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$ with 11 degrees of freedom.
5. Computations: $\bar{x} = 42$ kilowatt hours, $s = 11.9$ kilowatt hours, and $n = 12$. Hence,

$$t = \frac{42 - 46}{11.9/\sqrt{12}} = -1.16, \qquad P = P(T < -1.16) \approx 0.135.$$

6. Decision: Do not reject H_0 and conclude that the average number of kilowatt hours used annually by home vacuum cleaners is not significantly less than 46.

Two Samples: Tests on Two Means

The two-sided hypothesis on two means can be written generally as

$$H_0\colon \mu_1 - \mu_2 = d_0.$$

Obviously, the alternative can be two sided or one sided. Again, the distribution used is the distribution of the test statistic under H_0. Values $\bar{x}_1$ and $\bar{x}_2$ are computed and, for σ_1 and σ_2 known, the test statistic is given by

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}},$$

Unknown But Equal Variances

For the two-sided hypothesis

$$H_0\colon\ \mu_1 = \mu_2,$$
$$H_1\colon\ \mu_1 \neq \mu_2,$$

we reject H_0 at significance level α when the computed t-statistic

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{1/n_1 + 1/n_2}},$$

where

$$s_p^2 = \frac{s_1^2(n_1 - 1) + s_2^2(n_2 - 1)}{n_1 + n_2 - 2}$$

exceeds $t_{\alpha/2, n_1+n_2-2}$ or is less than $-t_{\alpha/2, n_1+n_2-2}$.

Unknown But Unequal Variances

There are situations where the analyst is **not** able to assume that $\sigma_1 = \sigma_2$. Recall from Section 9.8 that, if the populations are normal, the statistic

$$T' = \frac{(\bar{X}_1 - \bar{X}_2) - d_0}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$$

has an approximate t-distribution with approximate degrees of freedom

$$v = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{(s_1^2/n_1)^2/(n_1 - 1) + (s_2^2/n_2)^2/(n_2 - 1)}.$$

As a result, the test procedure is to *not reject* H_0 when

$$-t_{\alpha/2,v} < t' < t_{\alpha/2,v},$$

with v given as above. Again, as in the case of the pooled t-test, one-sided alternatives suggest one-sided critical regions.

Example 10.6

An experiment was performed to compare the abrasive wear of two different laminated materials. Twelve pieces of material 1 were tested by exposing each piece to a machine measuring wear. Ten pieces of material 2 were similarly tested. In each case, the depth of wear was observed. The samples of material 1 gave an average (coded) wear of 85 units with a sample standard deviation of 4, while the samples of material 2 gave an average of 81 with a sample standard deviation of 5. Can we conclude at the 0.05 level of significance that the abrasive wear of material 1 exceeds that of material 2 by more than 2 units? Assume the populations to be approximately normal with equal variances.

Solution

Let μ_1 and μ_2 represent the population means of the abrasive wear for material 1 and material 2, respectively.

1. H_0: $\mu_1 - \mu_2 = 2$.
2. H_1: $\mu_1 - \mu_2 > 2$.
3. $\alpha = 0.05$.
4. Critical region: $t > 1.725$, where $t = \frac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{1/n_1 + 1/n_2}}$ with $v = 20$ degrees of freedom.
5. Computations:

$$\bar{x}_1 = 85, \qquad s_1 = 4, \qquad n_1 = 12,$$
$$\bar{x}_2 = 81, \qquad s_2 = 5, \qquad n_2 = 10.$$

Hence

$$s_p = \sqrt{\frac{(11)(16) + (9)(25)}{12 + 10 - 2}} = 4.478,$$
$$t = \frac{(85 - 81) - 2}{4.478\sqrt{1/12 + 1/10}} = 1.04,$$
$$P = P(T > 1.04) \approx 0.16. \qquad \text{(See Table A.4.)}$$

6. Decision: Do not reject H_0. We are unable to conclude that the abrasive wear of material 1 exceeds that of material 2 by more than 2 units.

Tests Concerning Means

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H_0	Value of Test Statistic	H_1	Critical Region
$\mu = \mu_0$	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$; σ known	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$z < -z_\alpha$ $z > z_\alpha$ $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$
$\mu = \mu_0$	$t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$; $v = n - 1$, σ unknown	$\mu < \mu_0$ $\mu > \mu_0$ $\mu \neq \mu_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$z = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$; σ_1 and σ_2 known	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$z < -z_\alpha$ $z > z_\alpha$ $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$t = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{s_p\sqrt{1/n_1 + 1/n_2}}$; $v = n_1 + n_2 - 2$, $\sigma_1 = \sigma_2$ but unknown, $s_p^2 = \dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$t < -t_\alpha$ $t > t_\alpha$ $t < -t_{\alpha/2}$ or $t > t_{\alpha/2}$
$\mu_1 - \mu_2 = d_0$	$t' = \dfrac{(\bar{x}_1 - \bar{x}_2) - d_0}{\sqrt{s_1^2/n_1 + s_2^2/n_2}}$; $v = \dfrac{(s_1^2/n_1 + s_2^2/n_2)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$, $\sigma_1 \neq \sigma_2$ and unknown	$\mu_1 - \mu_2 < d_0$ $\mu_1 - \mu_2 > d_0$ $\mu_1 - \mu_2 \neq d_0$	$t' < -t_\alpha$ $t' > t_\alpha$ $t' < -t_{\alpha/2}$ or $t' > t_{\alpha/2}$